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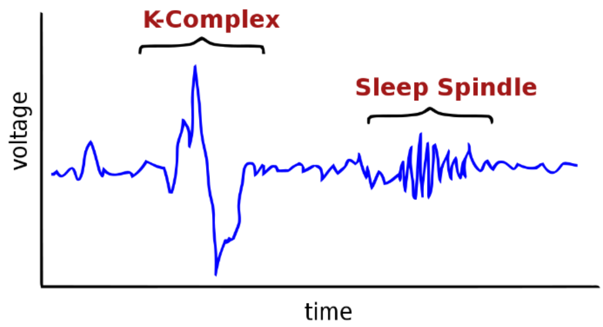
Frequency Domain II: Filters and Systems

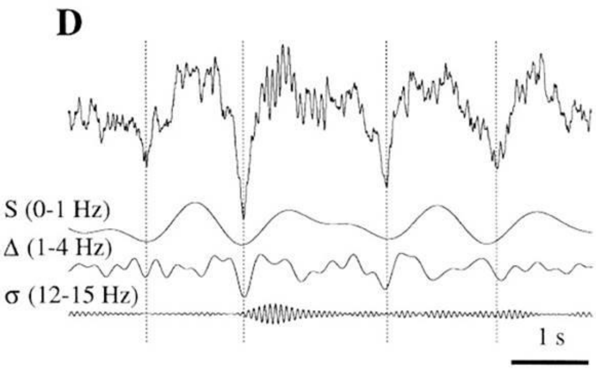
Special thanks: many of the following notes are based on Tal Dalal presentation for first degree SDA course.

Uses of filters in neuroscience studies

Commonly in sleep medical examination using EEG, we are interested in two main phenomena during stage 2 of NREM sleep- the K-complex and the sleep spindle. Those are considered the largest events in healthy human EEG during sleep.

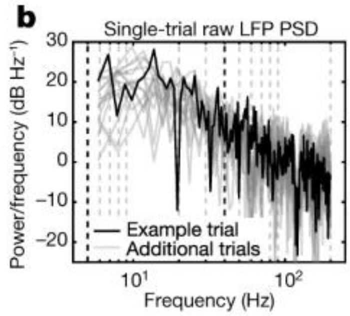


In practice, due to a lot of noise, those two phenomena are not always easy to see. Therefore, during the examination one can use filters, and then notice them (for example in the third row, frequency 12-15 Hz).



(Adopted from: https://biology.stackexchange.com/questions/44955/why-is-fast-fourier-transform-applied-to-raw-eeg-data)

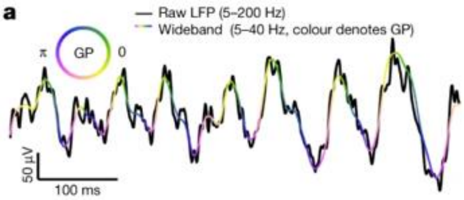
Nevertheless, not always the pre-defined filters are the ones that we want to use. For example, in davies et al. 2020, they were interested in better capturing the raw signal instead of filtering it. They created a wide band filter 5-40 hertz, based on the power spectrum of the raw LFP signal:



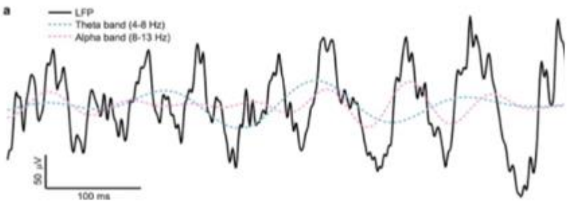
Using this filter, at any given window they found the dominant frequency and for which they computed its phase, called the global phase. As they demonstrate, this wide band filter allows them to better capture the waveform of the LFP, and produce a better spike-phase coupling:

GP analysis

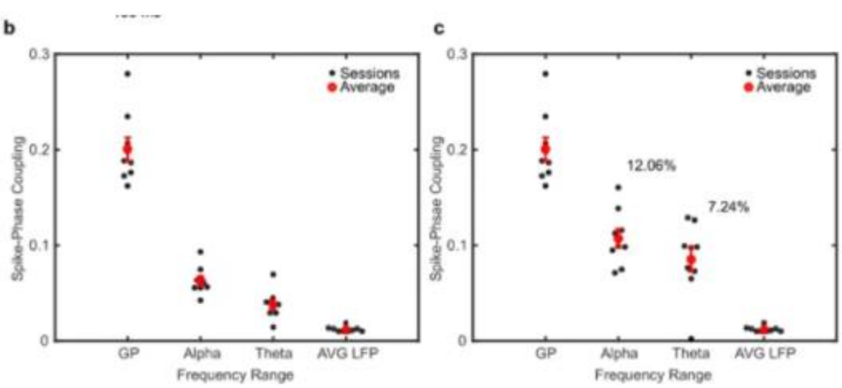
GP captures better the waveform of LFP



Narrow band analysis



Coupling to spike timing



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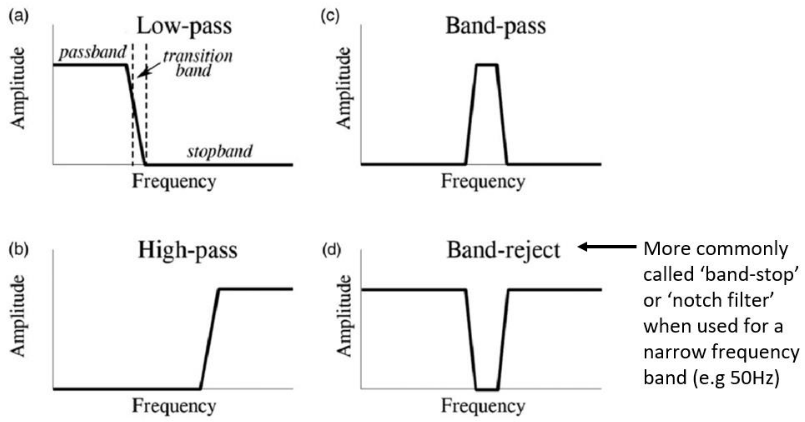
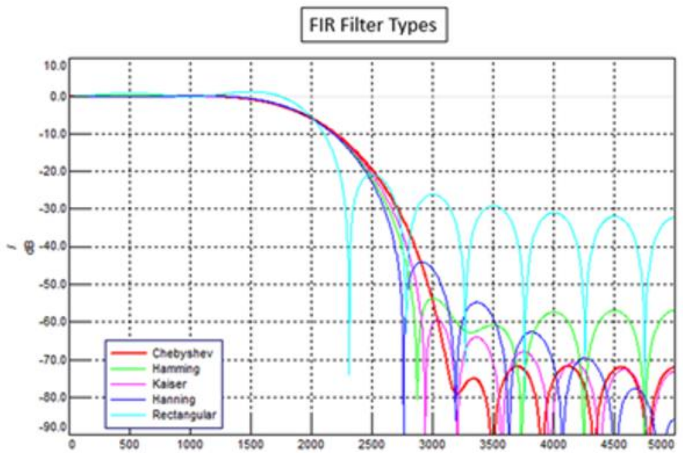
Types of filters

- A filter is a device or process that removes from a signal some unwanted component or feature, or to select a desired frequency range among many others.
- Filter can be either analogic or digital, we will focus on the latter.
- Filter in the time domain is equivalent to using a convolution in this dimension, therefore it is multiplication in the frequency domain.
- Filter classification:

  Spectral response: LPF, HPF, BPF, BSF, notch.



  Digital filters: FIR, IIR, Linear phase.



- Options/tools for filter construction:
  - Matlab code (firls, fir1, butter,…)
  - Matlab's filter visual tools (FDA tool (filterDesigner), wintool)
  - Fvtool – Filter viewing tool
  - Pythons scipy.signal module
  - PyFDA – Gui filter design tool

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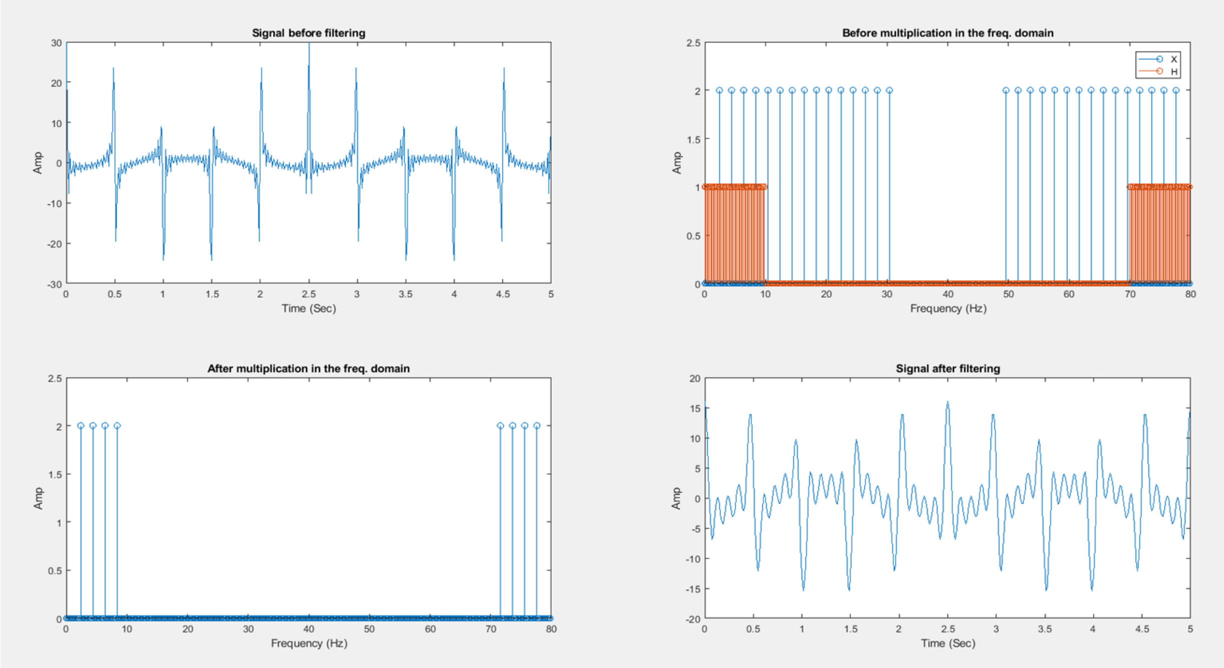
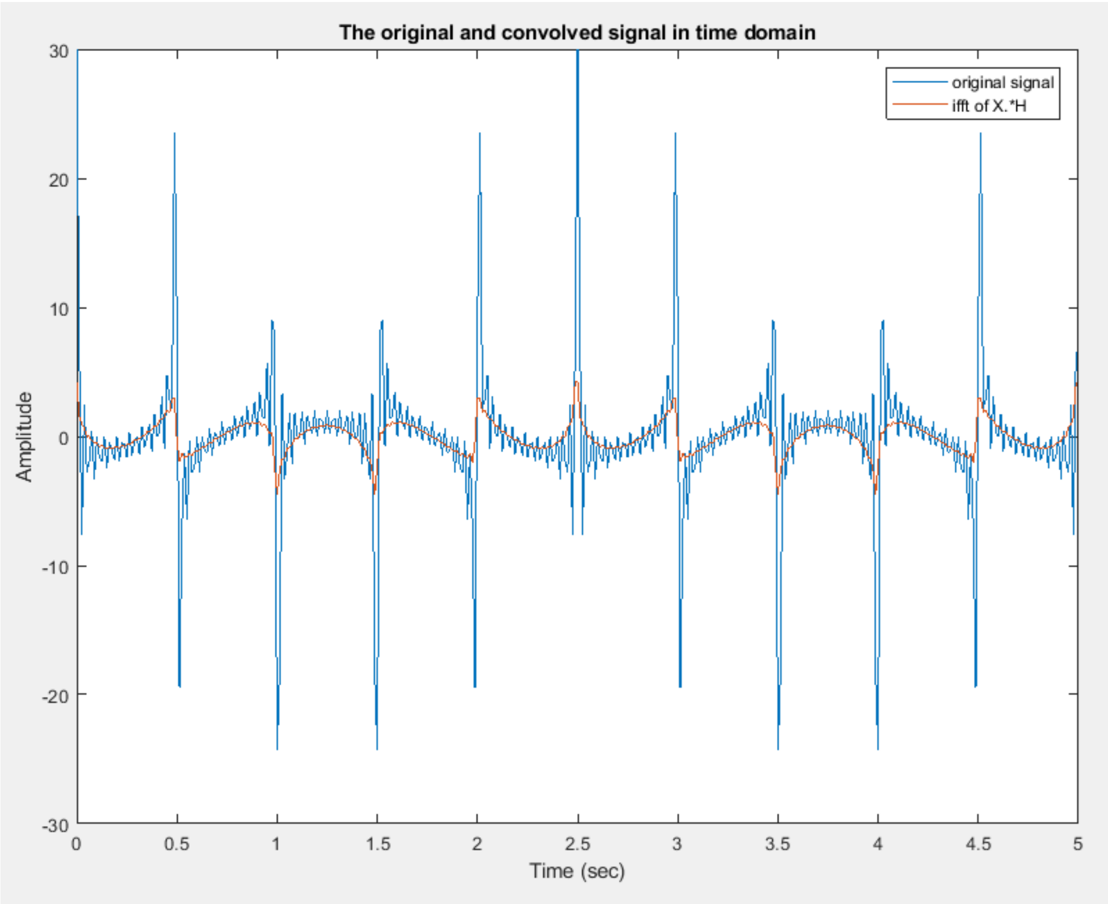
Example for LPF



Class Discussion: Demonstrate what frequencies were removed.



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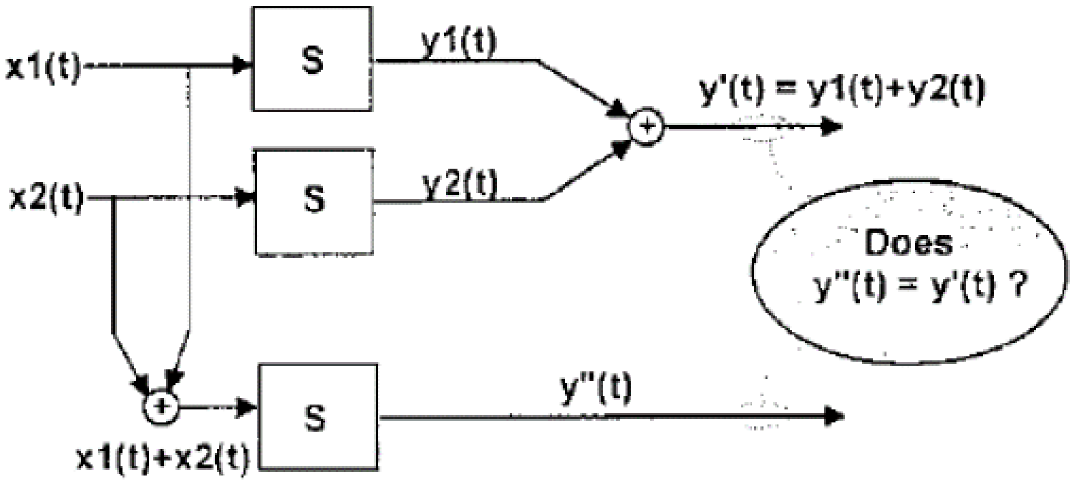
Systems

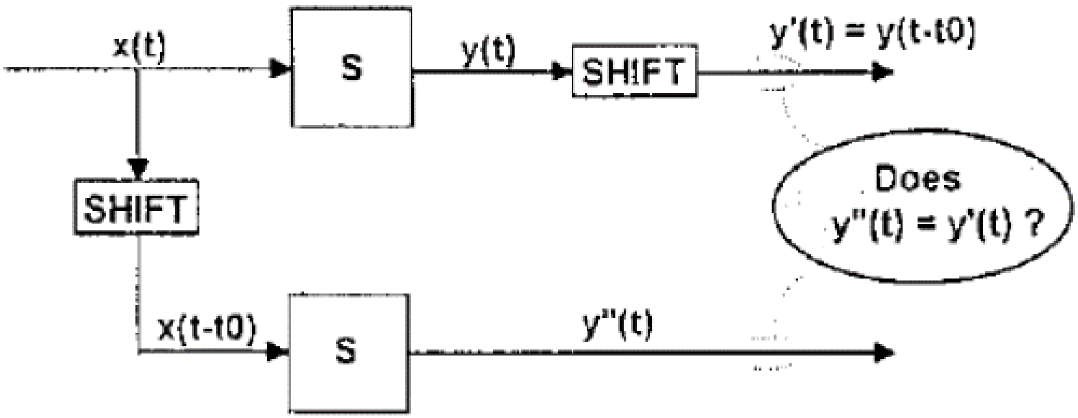
Linear systems fulfill two conditions:

Homogeneity: $af(x) = f(ax)$

Additivity: $f(x + y) = f(x) + f(y)$



Time invariant systems are systems in which the behavior of the system is fixed over time. For example, in the system $y(n) = x(n) + n$, for 2 different times: n=1 in the first and n=2 in the second, if $x(n) = 5$ than $y(n) = 6$ for the first one and $y(n) = 7$ for the second one, so in this case there is time dependency.



Class exercise: example from exam 2007

The amplifier neurons of the Levis Systemis function have the following response function: y(t)=2x(t). The neurons therefore act as a:

a. Linear system.
b. Time invariant system.
c. Linear time invariant (LTI) system.
d. None of the above.

Solution:

The system is both homogenic, additive and time independent. Therefore, this is a LTI system.

Impulse response:

Linear systems can be described as a linear multiplication of the input with the system's constant. We can describe it as a convolution of the input between two vectors (the input and the system constants). When we want to find the constants vector, we can use impulse response by entering an input of delta function and check the system behavior.

The impulse response defines two kinds of systems FIR and IIR when FIR can be represented by:

$$y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k]$$

In this system, at some time the impulse response is equal to zero. It might use "its memory", but it doesn't mean it does not equal to zero. This is a more simple and steady system, but its disadvantage is that this system required a lot of constants, so that the system is from high order.

On the contrary, IIR system can be represented by:

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

In this system, the impulse response is never equal to zero. This system is less steady (sometimes explodes). Nevertheless, its advantage is that its order is low, so that there is no need for many constants such as in FIR systems.

Basic examples to FIR and IIR systems:

- IIR oscillating impulse response: $y(n) = x(n) - y(n-1)$
  In this system the impulse response $x = 1,0,0,0,0,0,\ldots$ will give the output $y = 1,-1,1,-1,1,-1,\ldots$
- IIR to exploding impulse response: $y(n) = x(n) + 2y(n-1)$
  In this system the impulse response $x = 1,0,0,0,0,0,\ldots$ will give the output $y = 1,2,4,6,8,\ldots$
- FIR average last N samples: $y(n) = \sum_{k=0}^{N} \frac{x(n-k)}{N}$.
  In this system, when $N = 4$, the impulse response $x = 1,0,0,0,0,0,\ldots$ will give the output $y = 0.25,0.25,0.25,0.25,0,0,0,0,\ldots$

Class exercise: example from exam 2005

Draw the impulse response of a IIR filter defined by: $y(n) = 0.5y(n-1) + x(n)$

Calculate an FIR filter which will give equivalent output (with an impulse response error <10%).

Solution:

We will enter the impulse response $X = 1,0,0,0,0,\ldots$

So that its output is:

$I(1) = 0.5y(0) + x(1) = 1$

$I(2) = 0.5y(1) + x(2) = 0.5$

$I(3) = 0.5y(2) + x(3) = 0.25$

$I(4) = 0.5y(3) + x(4) = 0.125$

This is the series $I(n) = 0.5^{n-1}$, an infinite series in the form αr^k when $\alpha = 1,\ r = 0.5$, $k = n-1$. It's sum is $sum(y) = \frac{\alpha}{1-r} = \frac{1}{0.5} = 2$. Now we need to find FIR in which its sum is bigger then $2 \cdot (1-0.1) = 1.8$. We can check how many members from the series we need to take in order to describe the series. The sum of the for first members is 1.875, so that we can describe the system as FIR:

$$y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2) + \frac{1}{8}x(n-3)$$

Class exercise: example from exam 2007

The filter described by its impulse response $y(t) = x(t) + y(t-1)$:

a. Is a FIR filter. It is possible to create an equivalent IIR filter.

b. Is a FIR filter. It is impossible to create an equivalent IIR filter.

c. Is an IIR filter. It is possible to create an equivalent FIR filter.

d. Is an IIR filter. It is impossible to create an equivalent FIR filter.

Solution:

Due to the term $y(t-1)$, we understand this is an IIR filter. If we look at the impulse response, we can see that at each time point we get a sum of the current input with all the inputs so far, and therefore in order to create an FIR filter we will need infinite number of constants- therefore this filter explodes, and we can't create an equivalent FIR filter. Therefore, the answer is D.

FIR and IIR:

- Function like filters which operate on a given input.
- FIR is finite, therefore can relate to a finite number of inputs.
- IIR is infinite, therefore can relate recursively to itself.
- Commonly to understand what the filter does, we use impulse response, a deta function vector $X = [1,0,0,0,0,\ldots]$
- Basic examples to FIR and IIR systems:
  - IIR oscillating impulse response: $y(n) = x(n) - y(n-1)$
    In this system the impulse response $x = 1,0,0,0,0,0,\ldots$ will give the output $y = 1,-1,1,-1,1,-1,\ldots$
  - IIR to exploding impulse response: $y(n) = x(n) + 2y(n-1)$
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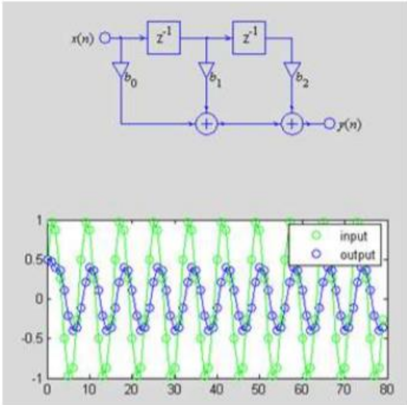
Example for FIR filter

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

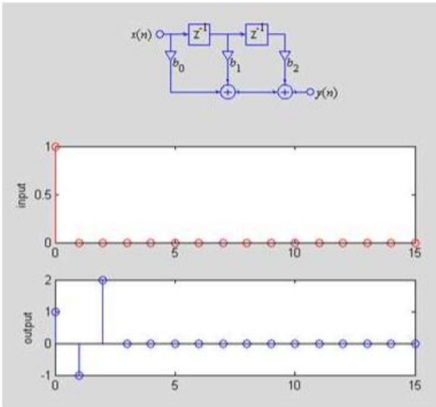
```
N = 80; k = 0:(N-1);
b0 = 1;
b1 = -1;
b2 = 1;
B = [b0 b1 b2];
f = 1/8;
x = sin(2*pi*f*k+pi/6);
y = filter(B,1,x);
```



Its impulse response:

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

```
N = 16; k = 0:(N-1);
b0 = 1;
b1 = -1;
b2 = 2;
B = [b0 b1 b2];
x = (k==0);
y = filter(B,1,x);
```

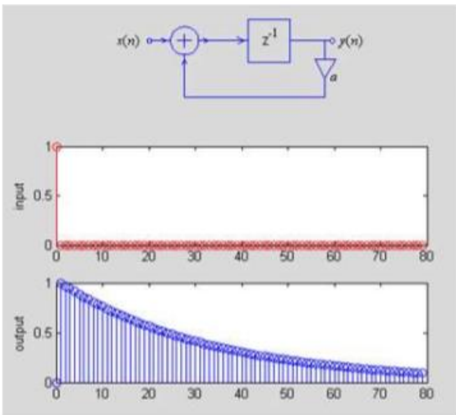


Example for IIR filters

Example 1:

$$y(n) = x(n-1) + a y(n-1)$$

```
N = 80; k = 0:(N-1);
a = 0.97;
B = [0 1];
A = [1 -a];
x = (k==0);
y = filter(B,A,x);
```



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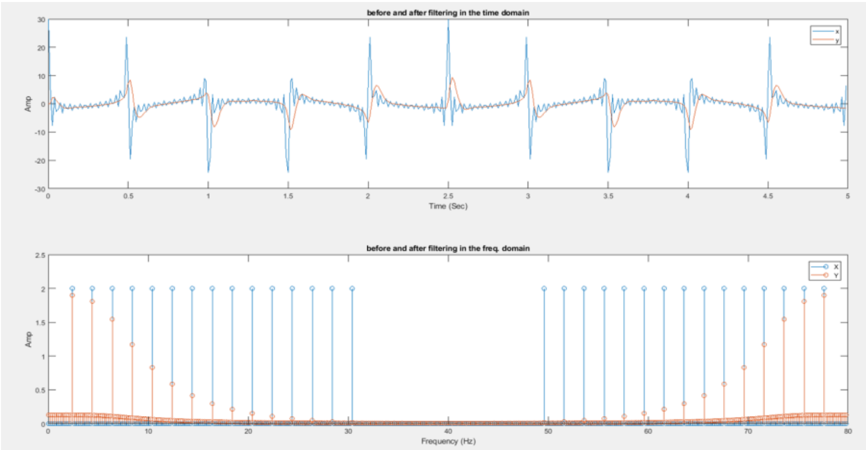
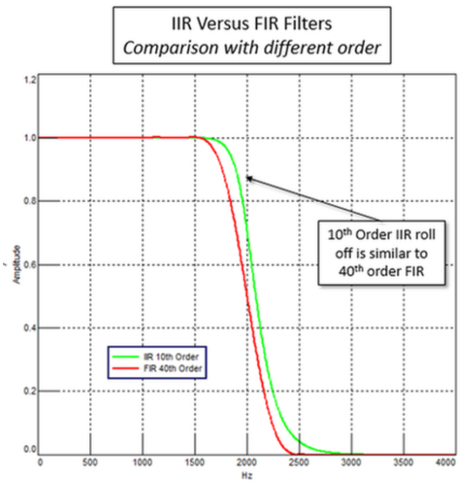
Example 2:

```
y(n)=0.065*x(n)+0.121*x(n-1)+0.0605*x(n-2)+1.194*y(n-1)-0.436*y(n-2);
```



Class Discussion: What is better- FIR or IIR?



IIR is faster

IIR is sharper (roll-off)

	IIR	FIR
Computational Speed	Fast – Low Order	Slow – High Order
Phase / Delay	Not constant	Constant
Stability	Sometimes	Always

Class Discussion: Can FIR and IIR be switched within each other?

Not necessarily, two examples:

Class exercise: example from exam 2005

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Solution:

We will enter the impulse response $X = 1{,}0{,}0{,}0{,}0, \ldots$

So that its output is: $I(1) = 0.5y(0) + x(1) = 1$ $\quad I(2) = 0.5y(1) + x(2) = 0.5$

$I(3) = 0.5y(2) + x(3) = 0.25$ $I(4) = 0.5y(3) + x(4) = 0.125$

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$$y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2) + \frac{1}{8}x(n-3)$$

Class exercise: example from exam 2007

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Solution:

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Class Exercises for self-learning:

example from exam 2006

The signal $V(t) = X \cdot \sin(20t \cdot 2\pi) + Y \cdot \cos(180t \cdot 2\pi)$ is sampled at 100 samples/sec. The sampled signal is then filtered using a 40Hz perfect high pass filter. The power spectrum of the sampled signal displays the following:

a. Single peak at 20Hz.
b. Two peaks at 20Hz & 180Hz.
c. Single peak at 180Hz.
d. No peaks in the spectrum.

Solution:

The signal was sampled using 100Hz, therefore there will be peaks in 20Hz and 80Hz instead of 20Hz and 180Hz due to aliasing. All along, because we used the sampling of 100Hz, we should see the window only between $\pm$50Hz, therefore only the 20 Hz is left.

Nevertheless, we used perfect HPF so we are left only with the range between 40 and 50 Hz, and therefore all other frequencies are eliminated and we are left without any peaks- answer d.

example from exam 2005

Two filters are given by the following equations:

(1) $y(n) = 2y(n-1) + x(n)$

(2) $y(n) = 8x(n-3) + 4x(n-2) + 2x(n-1) + x(n)$

a) Draw the impulse response of the two filters.

b) For each filter: is it FIR or IIR? Explain.

c) What is the output of the filters assuming a constant non-zero input? Explain.

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example from exam 2007

Two filters are given by the following equations:

(1) $y(n) = x(n) - y(n-1)$

(2) $y(n) = x(n) - x(n-1)$

a) Draw the impulse response of the two filters.

b) For each filter: is it FIR or IIR? Explain.

c) What is the output of the filters assuming a constant non-zero input? Explain.

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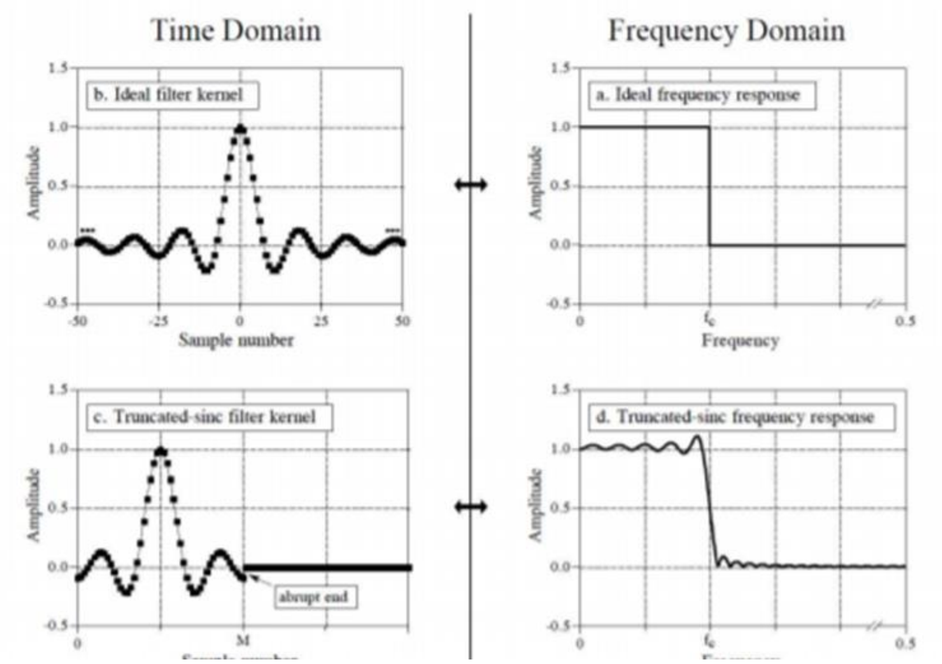
The ripples problem

One of Fourier transform essential limits, is that around discontinuity (step function), the Fourier function series are infinite sines and cosines. Those are commonly called the filter **ripples**. For this reason, filtering a specific band produces amplifications of other unwanted bands, due to the ripples.

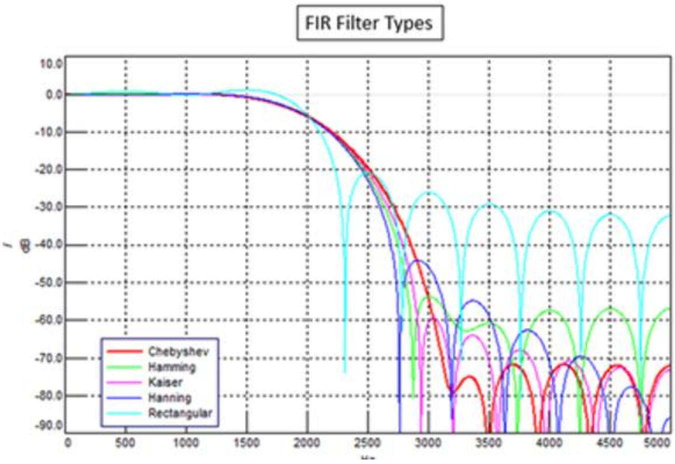
Class Discussion: What an ideal filter will look like in the frequency domain?

We will use the following illustrations to demonstrate the problems.



Class Discussion: How can we reduce ripples?

As we mentioned before, the problem is the discontinuity, therefore the sharp transition between the wanted frequencies and the unwanted. The technique in which we increase the transition is called Windowing, and is using FIR filters:



Class Discussion: How do we determine what filter to use?

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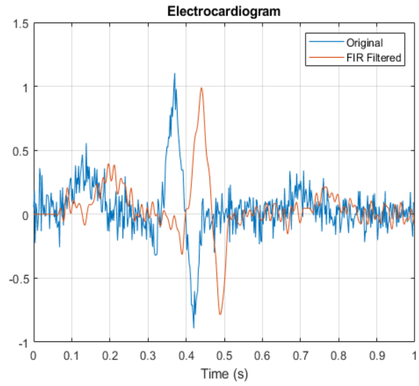
Problem solved? FIR are commonly can described not only as a function, but also as a kernel for a specific convolution multiplication (rectangle, triangle). For example the FIR $y(n) = \sum_{k=1}^{4} \frac{x(n-k)}{4}$ is the kernel of $[0.25, 0.25, 0.25, 0.25, 0, 0, 0, \ldots]$ as can be computed using the impulse response.

Nevertheless, the problem with FIR filters is that they tend to create a phase shift:



Luckily, the size of the shift for a symmetrical FIR filter is given by a simple equation. Given that the size of the filter is N than the size of the shift is $\frac{N-1}{2}$. For example if a signal was sampled with $60\ Hz$ and filtered by an FIR rectangle size of $N = 5$, then the filtered signal will be delayed by 2 samples which are $\frac{2}{60} = 0.033$ seconds.

The new phase can be computed by the equation $angle_{shift} = 2\pi k \cdot \frac{m}{N_{samples}}$ when k is the frequency being shifted, m is the number of samples shifted (we use minus if the shift is forward, as in the example above), $N_{samples}$ is the number of samples which can be computed by $f_s \cdot T$ (sample frequency and the length of the trial). If for example we have the same 1 sec signal from above, so for the frequency $k = 2Hz$ the new phase wil be: $angle_{shift} = 2\pi \cdot 2 \cdot \frac{-2}{60 \cdot 1} = -0.4819\ RAD$.

Practically, we commonly use the DC for shifting back the signal: